# Birationally rigid Fano varieties

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**Abstract**

We give a brief survey of the concept of birational rigidity, from its origins in the two-dimensional birational geometry, to its current state. The main ingredients of the method of maximal singularities are discussed. The principal results of the theory of birational rigidity of higher-dimensional Fano varieties and fibrations are given and certain natural conjectures are formulated.

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## 0. Introduction

This paper is based on the talk given by the author at the Fano conference in Turin. The aim of the paper is to give a brief survey of the concept of birational rigidity which nowadays is getting the status of one of the crucial concepts of higher-dimensional birational geometry. The Fano conference both by definition and its actual realization had a natural historical aspect. Therefore it seems most appropriate to review the story of birational rigidity, presenting the principal events in their real succession, from the first cautious steps made in XIX century to the modern rapid development. In this story Gino Fano himself played a prominent part: birational rigidity was one of his most important foresights.

For about fifty years Fano was the only mathematician in the world engaged in the field that in his time was a real terra incognita, three-dimensional birational geometry of algebraic varieties which are now called Fano varieties. He had a program of his own and he did his best to realize it, see [F1-F3]. On this way he discovered a lot of fascinating geometric constructions, found new approaches to investigating extremely deep and challenging problems, made a terrific amount of computations and guessed certain fundamental facts. He never completed his program. But even realization of a part of it took about thirty years of hard work of his successors, equipped with incomparably stronger techniques.

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The paper is an enlarged version of the talk, where some details and an explanation of the technique of hypertangent divisors, making the result of the paper [P] slightly stronger, are added. The present paper was written during my stay at Max-Planck-Institut für Mathematik in Bonn. I am very grateful to the Institute for the financial support, stimulating atmosphere and hospitality.

## 1. The Noether theorem

In [N] Max Noether published his famous theorem on the (two-dimensional) Cremona group: the group of birational self-maps of the (complex) projective plane

$$\operatorname{Bir}\mathbb{P}^2 = \operatorname{Cr}\mathbb{P}^2 = \operatorname{Aut}\mathbb{C}(s,t) = \{\chi\colon \mathbb{P}^2 - - \to \mathbb{P}^2\}$$

is generated by the group of projective automorphisms $\operatorname{Aut}\mathbb{P}^2$ and a single quadratic Cremona transformation which in a suitable coordinate system takes the form

$$\tau\colon (x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1). \tag{1}$$

His argument went as follows. Take an arbitrary birational self-map $\chi$ of $\mathbb{P}^2$ and consider the strict transform of the linear system of lines via $\chi^{-1}$:

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{\chi}{- - \rightarrow} & \mathbb{P}^2 \\ \left\{\begin{array}{c}\text{curves}\\ \text{of degree}\\ n \geq 1\end{array}\right\} & \longleftarrow & \{\text{lines}\} \\ \| & & \\ \left\{\begin{array}{c}\text{the linear}\\ \text{system } |\chi|\end{array}\right\} & & \end{array}$$

The moving (that is, free from fixed components) linear system $|\chi|$ becomes naturally the main subject of study. One has the following obvious alternative:

- either $n = 1$, in which case $\chi \in \operatorname{Aut} \mathbb{P}^2$ is regular,
- or $n \geq 2$, in which case $\chi$ is a birational map in the proper sense; in particular, the linear system $|\chi|$ has base points.

Assume that the second case holds. Since the curves in the linear system $|\chi|$ are rational and the *free intersection* (that is, the intersection outside the base locus) is equal to one, one can deduce that there exist at least three distinct points $o_1, o_2, o_3$ of the linear system $|\chi|$ satisfying the *Noether inequality*:

$$\sum_{i=1}^{3} \operatorname{mult}_{o_i} |\chi| > n.$$

If all three points $o_i$ lie on $\mathbb{P}^2$ (that is, there are no infinitely near points among them), then take the standard Cremona transformation $\tau$ (1) where $o_1, o_2, o_3$ are assumed to be the points

$$(1, 0, 0),\ (0, 1, 0, )\ \text{and}\ (0, 0, 1),$$

respectively. It is easy to compute that the linear system $|\chi \circ \tau|$ (which is the strict transform of $|\chi|$ via $\tau$, or the strict transform of the linear system of lines on $\mathbb{P}^2$ via the composition $\chi \circ \tau$) is a moving linear system of plane curves of degree

$$2n - \sum_{i=1}^{3} \operatorname{mult}_{o_i} |\chi| < n.$$

In other words, taking the composition with a quadratic transformation (which are all conjugate with each other by a projective automorphism), one can decrease the

degree $n \geq 1$. Thus (assuming that at each step there are no infinitely near points among $o_i$'s) we get a decomposition of $\chi$ into a product of quadratic transformations:

$$\chi = \tau_1 \circ \ldots \circ \tau_N \circ \alpha = \alpha_1 \circ \tau \circ \alpha_2 \circ \tau \circ \ldots \circ \tau \circ \alpha_{N+1},$$

where $\tau$ is the standard involution (1) and $\alpha, \alpha_i$ are all projective automorphisms.

There is an immense literature on the Noether theorem and Cremona transformations, see, for instance [H] (the book is to be soon re-published with an explanatory introduction written by V.A.Iskovskikh and M.Reid). Here we are interested only in the principal ingredients of Noether's argument. These are:

- the invariant $n \geq 1$ (the degree of curves in the linear system $|\chi|$),
- "maximal" triples of base points (that is, the triples satisfying the Noether inequality (1)),
- the "untwisting" procedure (decreasing $n$ and thus "simplifying" $\chi$).

**Remark.** One can well imagine that the untwisting procedure may not be uniquely determined. This is the case, when there are more than one "maximal" triples, so that we can decrease the degree $n$ taking the composition with various quadratic involutions. This naturally leads to *relations* between the generators of $\mathrm{Bir}\,\mathbb{P}^2$. They were first described by M.Gizatullin in [G]. Later the argument was radically simplified by V.A.Iskovskikh [I4].

## 2. Fano's work

At the beginning of the XXth century Fano started his work in three-dimensional birational geometry. He was an absolute pioneer in the field. His work lasted for about 50 years and, apart from its great mathematical value, presents an example of an equally great courage and inner strength.

Fano started with an attempt to reproduce Noether's argument in dimension three. His first object of study was the famous three-dimensional quartic $V = V_4 \subset \mathbb{P}^4$. His investigation went as follows. Take a birational self-map $\chi \in \mathrm{Bir}\,V$ and look at the strict transform of the linear system of hyperplane sections of $V$ via $\chi^{-1}$:

$$\begin{array}{ccc} V & \stackrel{\chi}{- - \rightarrow} & V \\ \left\{ \begin{array}{c} \text{a linear system} \\ \text{of divisors } |\chi| \\ \text{cut out on } V \text{ by} \\ \text{hypersurfaces} \\ \text{of degree } n \geq 1 \end{array} \right\} & \longleftarrow & \left\{ \begin{array}{c} \text{a linear system} \\ \text{of hyperplane} \\ \text{sections} \end{array} \right\} \end{array}$$

Now, similar to the two-dimensional case, we get an obvious alternative:

- either $n = 1$, in which case $\chi \in \operatorname{Aut} V$ is regular,
- or $n \geq 2$, in which case $\chi$ is a birational map in the proper sense; in particular, the linear system $|\chi|$ has a non-empty base locus.

According to the scheme of Noether's arguments, the next step to be made is finding a subscheme of high multiplicity in the base scheme of the linear system $|\chi|$. And indeed, Fano asserted that if $n \geq 2$, then one of the following possibilities holds:

- there exists a curve $B \subset V$ such that
$$\operatorname{mult}_B |\chi| > n, \tag{2}$$
- there exists a point $x \in V$ such that
$$\operatorname{mult}_x |\chi| > 2n, \tag{3}$$
- something similar happens, reminding of the unpleasant infinitely near case of the Noether theorem. Here Fano does not give any formal description, just presents an example of what can take place: there is a point $x \in V$ and an infinitely near line $L \subset E \cong \mathbb{P}^2$, where
$$\varphi \colon \widetilde{V} \to V$$
is the blowing up of $x$ with the exceptional divisor $E$, such that
$$\operatorname{mult}_x |\chi| + \operatorname{mult}_L \widetilde{|\chi|} > 3n,$$
where $\widetilde{|\chi|}$ is the strict transform of the linear system $|\chi|$ on $\widetilde{V}$.

Now Fano gives certain arguments, some of which are true and some not, showing that these cases are impossible. He concludes that the $n \geq 2$ case does not realize and therefore $n = 1$ is the only possible case. Thus

$$\operatorname{Bir} V = \operatorname{Aut} V.$$

Later Fano studied several other types of three-folds. One of his most impressive claims concerns the complete intersection

$$\begin{array}{ccc} V = & V_{2\cdot 3} & \subset \mathbb{P}^5 \\ & \| & \\ & F_2 \cap F_3 & \end{array}$$

of a quadric and a cubic in $\mathbb{P}^5$. Starting as above,

$$\begin{array}{ccc} V & \overset{\chi}{\dashrightarrow} & V \\ |nH| \supset |\chi| & \longleftarrow & \left\{ \begin{array}{c} \text{the linear system} \\ \text{of hyperplane} \\ \text{sections} \end{array} \right\} \end{array}$$

($H \in \operatorname{Pic} V$ is the class of a hyperplane section, $\operatorname{Pic} V = \mathbb{Z}H$), Fano discovers that for certain special subvarieties his inequality (2) can be satisfied. This is the case, when $B = L \subset V$ is a line. It is easy to see that the projection $V - - \to \mathbb{P}^3$ from the line $L$ is a dominant rational map of degree 2. Thus there exists a Galois involution

$$\tau_L \in \operatorname{Bir} V,$$

permuting points in a general fiber. One computes easily that

$$|\tau_L| \subset |4H| \quad \text{and} \quad \operatorname{mult}_L |\tau_L| = 5,$$

so that the linear system $|\tau_L|$ realizes the inequality (2). Now if

$$\operatorname{mult}_L |\chi| > n, \quad |\chi| \subset |nH|,$$

then one can check that the linear system $|\chi \circ \tau_L|$ is cut out on $V$ by hypersurfaces of degree

$$4n - 3 \operatorname{mult}_L |\chi| < n,$$

which gives the necessary "untwisting" procedure for $\chi$. The analogy to Noether's arguments is now complete. Let us once again look at the general scheme of Fano's arguments. They consist of the following components:

- the invariant $n$ ($|\chi| \subset |nH|$),
- existence of "maximal" curves or points (or something similar), satisfying the Fano inequalities (2,3),
- either excluding or untwisting the maximal curve or point found at the previous step (decreasing $n$ and "simplifying" $\chi$).

It should be added that the untwisting procedure is not always uniquely determined, that is, Fano inequalities are sometimes satisfied for a few various subvarieties, e.g. two different lines on $V_{2\cdot 3}$ (when the corresponding plane is contained in the quadric $F_2$). This naturally leads to relations between generators of $\operatorname{Bir} V$. For $V_{2\cdot 3}$ they were described by V.A.Iskovskikh around 1975, see [I3] and a detailed exposition in [IP].

## 3. The theorem of V.A.Iskovskikh and Yu.I.Manin

The modern birational geometry of three-dimensional varieties started in 1970 with two major breakthroughs: the theorem of H.Clemens and Ph.Griffiths on the three-dimensional cubic [CG] and the theorem of V.A.Iskovskikh and Yu.I.Manin on the three-dimensional quartic [IM]. In the latter paper Fano's ideas were developed into a rigorous and powerful theory, which made it possible to begin a systematic study of explicit birational geometry of three-folds. This success was to a considerable

degree prepared by the earlier papers of Yu.I.Manin on surfaces over non-closed fields: in [M1, M2] all the principal technical components of the method of maximal singularities were already present, including the crucial construction of the graph, associated with a finite sequence of blow ups.

Let us reproduce briefly the arguments of [IM]. Fix a smooth quartic $V = V_4 \subset \mathbb{P}^4$ and consider, as usual, the strict transform of the linear system of hyperplane sections with respect to $\chi^{-1}$, where $\chi\colon V - - \to V_4'$ is a birational map onto another smooth quartic:

$$\begin{array}{ccc} V & \overset{\chi}{- - \to} & V' \\ |nH| \supset |\chi| & \longleftarrow & \left\{ \begin{array}{c} \text{the linear system} \\ \text{of hyperplane} \\ \text{sections} \end{array} \right\} \end{array}$$

As usual, we come to the familiar alternative:

- either $n = 1$, in which case $\chi\colon V \to V'$ is an isomorphism (hence a projective isomorphism),
- or $n \geq 2$, in which case $\chi$ is a birational map in the proper sense; in particular, the base subscheme of the linear system $|\chi|$ is non-empty.

Assume that $n \geq 2$.

**Proposition 3.1.** *There exists a geometric discrete valuation $\nu$ on $V$ (here "geometric" means "realizable by a prime divisor $E$ on some model $\widetilde{V}$ of the field $\mathbb{C}(V)$") such that the inequality*

$$\nu(\Sigma) > n \cdot \text{discrepancy}(\nu) \tag{4}$$

*holds, where $\nu(|\chi|) = \nu(D)$ for a general divisor $D \in |\chi|$.*

The discrete valuations $\nu$, satisfying (4), are called *maximal singularities* of the linear system $|\chi|$. The inequality (4) is called the *Noether-Fano inequality*.

Now [IM] shows that a moving linear system $|\chi|$ on $V$ cannot have a maximal singularity. The hardest case is when the centre of the maximal singularity $\nu$ is a point: centre$(E) = x \in V$. In this case take two general divisors $D_1, D_2 \in |\chi|$ and consider the cycle of scheme-theoretic intersection

$$Z = (D_1 \circ D_2).$$

It is an effective curve on $V$. The crucial fact is given by

**Proposition 3.2.** *The following inequality holds*

$$\text{mult}_x Z > 4n^2.$$

Since $\deg Z = 4n^2$, this gives a contradiction. Thus we obtain

**Theorem 3.1.[IM]** *Any birational map between smooth three-dimensional quartics is a projective isomorphism.*

However this very argument gives immediately a much stronger claim! For instance, let us describe birational maps

$$\begin{array}{ccc} V & \stackrel{\chi}{- - \to} & V' \\ & & \downarrow \; \pi' \\ & & S' \end{array}$$

from $V$ to conic bundles $V'/S'$ (or, in a slightly different language, describe the *conic bundle structures* on $V$). Let us construct a moving linear system $\Sigma'$ on $V'$ in the following way:

$$\begin{array}{cccc} & V' & \Sigma' & \\ & & & \\ \pi' & \downarrow & \uparrow & \text{pull back via } \pi' \\ & S' & \left\{ \begin{array}{c} \text{moving linear} \\ \text{system of curves} \\ \text{on the surface } S' \end{array} \right\} & \end{array}$$

Consider the strict transform $\Sigma \subset |nH|$ of $\Sigma'$ on $V$ via $\chi$. Then it is easy to prove that a maximal singularity exists *always*, now irrespective of the value of $n \geq 1$. However, we know that existence of a maximal singularity leads to a contradiction. Therefore, the birational map $\chi$ simply cannot exist!

What was actually proved in [IM], can be formulated as follows:

*a smooth three-dimensional quartic $V \subset \mathbb{P}^4$*

- *cannot be fibered into rational curves by a rational map,*
- *cannot be fibered into rational surfaces by a rational map,*
- *if $\chi\colon V - - \to V'$ is a birational map, where $V'$ is a $\mathbb{Q}$-Fano threefold, then $\chi\colon V \to V'$ is an isomorphism.*

Speaking the modern language, we express all this by saying that the quartic is *birationally superrigid*.

## 4. The method of maximal singularities

In all the procedures described above a certain integral parameter was involved — namely, the "degree" $n$ of the linear system $\Sigma$, defining the birational map $\chi$

under consideration. All the above examples dealt with Fano varieties $V$ such that $\mathrm{Pic}\, V \cong \mathbb{Z}$, so that the integer $n$ meant just the class of $\Sigma$ in $\mathrm{Pic}\, V$. However, this extremely important number has a more general invariant meaning, which we describe now.

Let $X$ be a uniruled $\mathbb{Q}$-Gorenstein variety with terminal singularities. This assumption implies that the canonical class $K_X$ is negative on some family of (generically irreducible) curves sweeping out $X$. Therefore for any divisor $D$ the following number is finite:

$$c(D, X) = \sup\{b/a | b, a \in \mathbb{Z}_+ \setminus \{0\}, |aD + bK_X| \neq \emptyset\}.$$

It is called the *threshold of canonical adjunction* of the divisor $D$. Sometimes we omit $X$ and write simply $c(D)$ or $c(\Sigma)$ for $D \in \Sigma$ moving in a linear system.

Now we can describe the general scheme of the method of maximal singularities. Let us fix a uniruled variety $V$ with $\mathbb{Q}$-factorial terminal singularities and another variety $V'$ in this class. Let us assume that $V'$ is birational to $V$. The aim of the method is to give a complete description of all possible birational correspondences between $V$ and $V'$.

We start as usual with the following diagram:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \rightarrow} & V' & \\ \text{moving linear system} & \Sigma & \longleftarrow & \Sigma' & \text{moving linear system} \\ & c(\Sigma) & ? & c(\Sigma') & \end{array}$$

The linear system $\Sigma'$ is fixed throughout the whole argument. The thresholds are taken with respect to the varieties $V$, $V'$. The ? sign means that we do not know, which inequality is true: "$\leq$" or "$>$".

Now we get the alternative:

- either ? is $\leq$, in which case we stop. It is presumed that when we have the inequality
$$c(\Sigma) \leq c(\Sigma'), \tag{5}$$
then "we can say everything" about the map $\chi$. In real life, sometimes this is the case, sometimes not. But in any case this inequality completely reduces the birational problem to a biregular one, since the family of linear systems $\Sigma$ is bounded (in many cases (5) implies that it is actually empty or $\Sigma$ is unique, as above).

- or ? is $>$, in which case we proceed further as follows.

**Proposition 4.1.** *There exists a geometric discrete valuation $\nu = \nu_E$ on $V$ such that the inequality*
$$\nu(\Sigma) > c(\Sigma) \cdot a(E) \tag{6}$$

*holds.*

The inequality (6) is called *the Noether-Fano inequality.* The discrete valuation $\nu$ is called a *maximal singularity* of the linear system $\Sigma$.

The word "geometric" means, as we have mentioned above, that there exists a birational morphism $\varphi\colon \widetilde{V} \to V$ with $\widetilde{V}$ smooth such that $\nu = \nu_E$ for some prime divisor $E \subset \widetilde{V}$. In (6) $\nu_E(\Sigma)$ means the multiplicity of a general divisor $D \in \Sigma$ at $E$ and $a(E)$ means the discrepancy of $E$.

Now for *each* geometric discrete valuation $E$ the following work should be performed:

- either $E$ can be excluded as a possible maximal singularity: there is no *moving* linear system $\Sigma$ satisfying the Noether-Fano inequality (6) for $E$,
- or $E$ should be untwisted. The untwisting means that we find a birational self-map
$$\chi_E^* \in \operatorname{Bir} V$$
such that we get the following diagram:
$$\begin{array}{ccc} V & \stackrel{\chi_E^*}{-\,-\rightarrow} & V \\ \Sigma^* & \longleftarrow & \Sigma \\ c(\Sigma^*) & < & c(\Sigma). \end{array}$$
Replacing $\chi$ by
$$\chi \circ \chi_E^*\colon V - - \rightarrow V',$$
we go back to the beginning of the procedure (compare $c(\Sigma^*)$ and $c(\Sigma')$ and so on).

## 5. Birationally rigid varieties

Roughly speaking $V$ is said to be *birationally rigid*, if the above-described procedure works for $V$, that is, in a finite number of steps we obtain the desired inequality (5).

**Definition 5.1.** $V$ is said to be birationally rigid, if for *any* $V'$, *any* birational map $\chi\colon V - - \rightarrow V'$ and *any* moving linear system $\Sigma'$ on $V'$ there exists a birational self-map $\chi^* \in \operatorname{Bir} V$ such that the following diagram holds:
$$\begin{array}{ccccc} V & \stackrel{\chi^*}{-\,-\rightarrow} & V & \stackrel{\chi}{-\,-\rightarrow} & V' \\ \Sigma^* & \longleftarrow & \Sigma & \longleftarrow & \Sigma' \\ c(\Sigma^*) & & \leq & & c(\Sigma') \end{array}$$

The birational self-map $\chi^*$ is a composition of elementary untwisting maps described above:

$$\chi^* = \chi^*_{E_1} \circ \ldots \circ \chi^*_{E_N}.$$

However, it turns out that for many (hopefully, for "majority" of) Fano varieties the untwisting procedure is trivial.

**Definition 5.2.** $V$ is said to be birationally superrigid, if for *any* $V'$, *any* birational map $V - - \to V'$ and *any* moving linear system $\Sigma'$ on $V'$ the following diagram holds:

$$\begin{array}{ccc} V & \stackrel{\chi}{- - \to} & V' \\ & & \\ \Sigma & \longleftarrow & \Sigma' \\ & & \\ c(\Sigma) & < & c(\Sigma'). \end{array}$$

In other words, a birationally rigid variety is superrigid, if we may always take $\chi^* = \mathrm{id}_V$. In the sense of the given definitions, the smooth quartic $V_4 \subset \mathbb{P}^4$ is superrigid and the smooth complete intersection $V_{2\cdot 3} \subset \mathbb{P}^5$ is rigid (for the latter case, the proof has been so far produced for a generic member of the family only, see [I3,P2,IP]).

Immediate geometric implications of birational (super)rigidity, which actually determined the very choice of this word combination, are collected below.

**Proposition 5.1.** *Let $V$ be a smooth Fano variety with* $\mathrm{Pic}\, V = \mathbb{Z}K_V$. *If $V$ is birationally rigid, then*

- *$V$ cannot be fibered into rationally connected (or uniruled) varieties by a non-trivial rational map; that is, the following diagram is impossible*

  $$\begin{array}{ccl} V & - - \to & V' \\ & & \downarrow \quad \text{uniruled fibers} \\ & & S' \end{array}$$

  *with* $\dim S' \geq 1$,

- *if $\chi\colon V - - \to V'$ is a birational map onto a $\mathbb{Q}$-Fano variety with* $\mathrm{rk}\,\mathrm{Pic}\, V' = 1$, *then*

  $$V \cong V'$$

  *(although $\chi$ itself may be not a biregular map). If, moreover, $V$ is superrigid, then $\chi$ itself is an isomorphism. In particular, in the superrigid case the groups of birational and biregular self-maps coincide,*

  $$\mathrm{Bir}\, V = \mathrm{Aut}\, V. \tag{7}$$

  *Conversely, if $V$ is rigid and (7) holds, then by definition of rigidity it is clear that $V$ is superrigid.*

The known examples motivate the following

**Conjecture 5.1.** *Let $V$ be a smooth Fano variety of dimension $\dim V \geq 4$ with $\mathrm{Pic}\, V = \mathbb{Z}K_V$. Then $V$ is birationally rigid. If $\dim V \geq 5$, then $V$ is superrigid.*

The biggest class of higher-dimensional Fano varieties, supporting this conjecture, is given by the following

**Theorem 5.1.** **[P5,P8]** *Let*

$$V = \begin{array}{c} V_{d_1 \cdot d_2 \cdot \ldots \cdot d_k} \\ \| \\ F_1 \cap F_2 \cap \ldots \cap F_k \end{array} \subset \mathbb{P}^{M+k}$$

*be a sufficiently general (in the sense of Zariski topology) Fano complete intersection of the type $d_1 \cdot d_2 \cdot \ldots \cdot d_k$, where $d_1 + \ldots + d_k = M + k$, $M \geq 4$ and $2k < M = \dim V$. Then $V$ is birationally superrigid.*

More examples of superrigid Fano varieties are given by smooth complete intersections in weighted projective spaces, see [P6].

**Remark.** As A.Beauville kindly informed the author, in dimension 4 there is still a gap between rigidity and superrigidity. His example is a $2 \cdot 2 \cdot 3$ smooth complete intersection in $\mathbb{P}^7$, containing a two-dimensional plane $P \cong \mathbb{P}^2$. There are no reasons to doubt their rigidity (although it has not yet been proved). However, for these special complete intersections

$$\mathrm{Bir}\, V \neq \mathrm{Aut}\, V,$$

since the projection from the plane $P$,

$$\pi \colon V - - \to \mathbb{P}^4,$$

is of degree two and therefore generates a Galois involution $\tau \in \mathrm{Bir}\, V$ which is not regular. Therefore such varieties are not superrigid. In dimension $\geq 5$ constructions of this type are not possible.

Most of the proofs of birational superrigidity make use of the following sufficient condition.

**Theorem 5.2.** *Let $X$ be a smooth Fano variety with $\mathrm{Pic}\, X = \mathbb{Z}K_X$. Assume that for any irreducible subvariety $Y \subset X$ of codimension two the following two properties are satisfied:*

(i) $\mathrm{mult}_Y \Sigma \leq n$ *for any linear system $\Sigma \subset |-nK_X|$ without fixed components*;

(ii) *the inequality*

$$\mathrm{mult}_x Y \leq \frac{4}{\deg X} \deg Y \tag{8}$$

*holds for any point* $x \in Y$, *where*

$$\deg X = (-K_X)^{\dim X}, \quad \deg Y = (Y \cdot (-K_X)^{\dim Y})$$

*and* $\mathrm{mult}_Y \Sigma$ *means multiplicity of a general divisor* $D \in \Sigma$ *along* $Y$.

*Then the variety* $X$ *is birationally superrigid.*

The strongest technique which makes it possible to check the (principal) condition (ii) of this criterion is that of *hypertangent divisors*. Although at the moment an alternative method, based on the connectedness principle of Shokurov and Kollár [Sh,K] (suggested by Corti [C2] and later used in [CM] and [P10]), gains momentum, the older argument by hypertangent divisors is still working better. For the reader to get the idea of this technique, we give here a proof for Fano hypersurfaces $V = V_M \subset \mathbb{P}^M$. Birational superrigidity of any smooth hypersurface has already been proved in [P10]. Nevertheless we give here an argument which is based on the paper [P5], slightly sharpening the result.

**Proposition 5.2.** *Let* $x \in V = V_M \subset \mathbb{P}^M$ *be a point such that there are but finitely many lines* $L \subset V$ *passing through* $x$. *Then for any irreducible subvariety* $Y \subset V$ *of codimension two the estimate*

$$\frac{\mathrm{mult}_x}{\deg} Y \leq \frac{4}{\deg V} = \frac{4}{M} \tag{9}$$

*holds.*

**Proof.** Let $(z_1, \dots, z_M)$ be a system of affine coordinates on $\mathbb{P} = \mathbb{P}^M$ with the origin at $x$. Write down the equation of the hypersurface $V$:

$$f = q_1 + q_2 + \ldots + q_M,$$

where $q_i$ are homogeneous of degree $i$ in $z_*$. Note that the lines through $x$ on the hypersurface $V$ are given by the system of equations

$$q_1 = q_2 = \ldots = q_M = 0. \tag{10}$$

Therefore the set (10) of common zeros is of dimension at most one. Denote by

$$f_i = q_1 + \ldots + q_i$$

the truncated polynomials. It is clear that in the affine open set $\mathbb{A} = \mathbb{A}^M_{(z_1,\dots,z_M)} \subset \mathbb{P}$ the algebraic set

$$f_1 = f_2 = \ldots = f_M = 0$$

is the same as (10), therefore it is of dimension at most one. This implies, in its turn, that the algebraic set

$$f_1|_{\mathbb{A}\cap V} = f_2|_{\mathbb{A}\cap V} = \ldots = f_{M-1}|_{\mathbb{A}\cap V} = 0$$

on the affine part of the hypersurface $V$ is also of dimension at most one: scheme-theoretically it is the same as (10), supported on the union of lines on $V$ through $x$.

Let us look at the divisors

$$D_i = \overline{\{f_i|_{\mathbb{A}\cap V} = 0\}},$$

$i = 1, \ldots, M-1$. We call them *hypertangent divisors*: if $H \in \operatorname{Pic} V$ is the class of a hyperplane section, then clearly

$$D_i \in |iH|$$

and

$$\operatorname{mult}_x D_i \geq i+1,$$

since in the affine part of $V$

$$D_i|_{\mathbb{A}\cap V} = \{(q_{i+1} + \ldots + q_M)|_V = 0\}.$$

Now by assumption

$$\dim_x(D_1 \cap \ldots \cap D_{M-1}) \leq 1,$$

where $\dim_x$ means the dimension in a neighborhood of the point $x$. It is easy to see that for a given subvariety $Y \ni x$ of codimension two there is a set of $(M-4)$ divisors

$$\{D_i \,|\, i \in \mathcal{I}\} \subset \{D_1, \ldots, D_{M-1}\}$$

such that

$$\dim_x \left( Y \cap \bigcap_{i\in\mathcal{I}} D_i \right) = 1.$$

Now let us order the set $\mathcal{I}$ somehow, so that

$$\{D_i \,|\, i \in \mathcal{I}\} = \{R_1, R_2, \ldots, R_{M-4}\}.$$

It is easy to construct by induction on $i \in \{0, \ldots, M-4\}$ the sequence of irreducible subvarieties

$$Y_0 = Y, Y_1, \ldots, Y_{M-4},$$

such that

- $Y_{i+1} \subset Y_i$, $\operatorname{codim} Y_i = i+2$ (the codimension is taken with respect to $V$);
- $Y_i \not\subset R_{i+1}$, so that $(Y_i \circ R_{i+1})$ is an effective algebraic cycle on $V$, $Y_{i+1}$ is one of its irreducible components;
- the following estimate holds:

$$\frac{\operatorname{mult}_x}{\deg} Y_{i+1} \geq \frac{\operatorname{mult}_x}{\deg} Y_i \cdot \frac{\operatorname{mult}_x R_{i+1}}{a_{i+1}},$$

  where $R_j \in |a_j H|$, $j = 1, \ldots, M-4$.

Now $C = Y_{M-4}$ is an irreducible curve on $V$, satisfying the inequality

$$\frac{\mathrm{mult}_x}{\deg} C \geq \frac{\mathrm{mult}_x}{\deg} Y \cdot \frac{5}{4} \cdot \dots \cdot \frac{M}{M-1}. \tag{11}$$

(If the set $\mathcal{I} \neq \{4, \dots, M-1\}$, then the estimate is better: we take the worst possible case.) Making the obvious cancellations and taking into consideration that the left-hand side of (11) cannot exceed 1, we obtain the desired estimate (9).

**Remark.** When we define informally birationally rigid varieties as those for which the method of maximal singularities works, one may naturally ask, what happens if it does not. There is an answer in dimension three. It is given by the Sarkisov program, developed by V.G.Sarkisov (see [S3]) and completely proved by Corti in [C1]. The answer is, that when it is possible neither to exclude nor to untwist a maximal singularity, it should be eliminated by a link to another Mori fiber space. We do not touch these points in the present paper. See [S3,R,C1,C2,CR,CM] for the details.

## 6. Singular Fano varieties

Up to this moment, all our examples dealt with smooth Fano varieties. Here we give the most interesting cases of birationally rigid Fano varieties with isolated terminal singularities. The oldest example of a birationally rigid singular Fano 3-fold is the three-dimensional quartic with a unique non-degenerate double point [P1].

Let $x \in V = V_4 \subset \mathbb{P}^4$ be the singularity. There are 24 lines on $V$ passing through $x$; denote them by $L_1, \dots L_{24}$. With the point $x \in V$ a birational involution $\tau \in \mathrm{Bir}\, V$ is naturally associated: the projection from $x$

$$\begin{array}{rcccl} & V & \subset & \mathbb{P}^4 & \\ \text{rational map} & | & & | & \text{rational map} \\ \text{of degree 2} & | & & | & \text{with the fiber } \mathbb{P}^1 \\ & \downarrow & & \downarrow & \\ & \mathbb{P}^3 & = & \mathbb{P}^3 & \end{array}$$

determines the Galois involution $\tau$ of $V$ over $\mathbb{P}^3$.

Let $L = L_i$ be a line through $x$. Look at the projecttion from the line $L$:

$$\begin{array}{rcccl} & V & \subset & \mathbb{P}^4 & \\ \text{rational map} & | & & | & \text{rational map} \\ \text{with the fibers —} & | & & | & \text{with the fiber } \mathbb{P}^2 \\ \text{cubic curves} & \downarrow & & \downarrow & \\ & \mathbb{P}^2 & = & \mathbb{P}^2. & \end{array}$$

Thus $V$ is fibered over $\mathbb{P}^2$ into elliptic curves. Since $V$ is singular at $x$, all the cubic curves pass through $x$, which means that the fibration $V/\mathbb{P}^2$ has a section. Taking

it for the zero of a group law on a generic fiber, we get the birational involution

$$\tau_i = \tau_L \in \operatorname{Bir} V,$$

the reflection from zero on a generic fiber. Set $\tau_0 = \tau$. Let $B(V)$ be the subgroup of $\operatorname{Bir} V$, generated by the involutions $\tau_0, \tau_1, \ldots, \tau_{25}$.

**Theorem 6.1. [P1]** (i) *$V$ is birationally rigid.*
(ii) *The group $B(V)$ is the free product of 25 cyclic subgroups $\langle \tau_i \rangle = \mathbb{Z}/2\mathbb{Z}$, where $i = 0, 1, \ldots, 25$:*

$$B(V) = \mathop{\ast}_{i=0}^{24} \langle \tau_i \rangle.$$

(iii) *The subgroup $B(V) \subset \operatorname{Bir} V$ is normal and the following exact sequence takes place:*

$$1 \to B(V) \to \operatorname{Bir} V \to \operatorname{Aut} V \to 1.$$

First proved in [P1], this theorem was later discussed by Corti in [C2], where due to an application of the connectedness principle of Shokurov and Kollár [Sh,K] the proof was simplified. The further study of singular quartics was performed in [CM].

Theorem 6.1 can be generalized in higher dimensions [P9]. Let $V = V_M \subset \mathbb{P}^M$, $M \geq 5$, be a sufficiently general (for the precise conditions see [P9]) hypersurface with isolated terminal singularities. For a singular point $x \in V$ we obviously have

$$\mu_x = \operatorname{mult}_x V \leq M - 2,$$

and if $\mu_x = M - 2$, then the conditions of general position imply that there is only one point with this multiplicity. Let us define the integer

$$\mu = \max_{x \in \operatorname{Sing} V} \{\mu_x\} \leq M - 2.$$

**Theorem 6.2.** (i) *Sufficiently general variety $V$ is birationally rigid.*
(ii) *If $\mu \leq M - 3$, then $V$ is superrigid.*
(iii) *If $\mu = M - 2$ and $\mu_x = \mu$ for the unique point $x$, then*

$$\operatorname{Bir} V = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z},$$

*where $\tau \in \operatorname{Bir} V$ is the Galois involution determined by the rational map*

$$\begin{array}{rcccl} & V & \subset & \mathbb{P}^M & \\ \text{rational map} & | & & | & \text{projection} \\ \text{of degree 2} & | & & | & \text{from } x \\ & \downarrow & & \downarrow & \\ & \mathbb{P}^{M-1} & = & \mathbb{P}^{M-1}. & \end{array}$$

Of course, singular Fano varieties are much more numerous in types than the smooth ones. The smooth quartics generalize to 95 families of $\mathbb{Q}$-Fano hypersurfaces

$$V = V_d \subset \mathbb{P}(1, a_1, \ldots, a_4),$$

$a_1 + a_2 + a_3 + a_4 = d$. They all have terminal quotient singularities but in a sense are closer by their properties to smooth Fano varieties.

**Theorem 6.3.** **[CPR]** *A general member $V$ of each of 95 families is birationally rigid. The group of birational self-maps is generated by finitely many involutions.*

For each of 95 families these involutions were explicitly described in [CPR]. This paper is based on the classical method of maximal singularities combined with the Sarkisov program [S3,R,C1].

## 7. The relative version

So far we have been considering the absolute case, that is, the case of Fano varieties. However, the world of rationally connected varieties is much bigger. If we assume the predictions of the minimal model program, each rationally connected variety is birational to a fibration into Fano varieties over a base that, generally speaking, is not necessarily a point.

Here we give a very brief outline of the relative version of the rigidity theory — that is, rigidity theory of non-trivial fibrations. Similar to the absolute case of Fano varieties, the starting point here was formed by "two-dimensional Fano fibrations" over a non-closed field, or simply speaking, surfaces with a pencil of rational curves over a non-closed field. In the papers of V.A.Iskovskikh [I1,I2] (which continued the work started in the papers of Yu.I.Manin on del Pezzo surfaces over non-closed fields [M1,M2], see also [M3]) it was proved that under certain conditions there is only one pencil of rational curves. This theorem was the first relative rigidity result. It was necessary to generalize these claims and, in the first place, the technique of the proof to higher dimensions.

This breakthrough was made by V.G.Sarkisov in [S1,S2]. Let us consider smooth conic bundles of dimension $\geq 3$:

$$\begin{array}{ccccccc} & & V & \hookrightarrow & \mathbb{P}(\mathcal{E}) & & \\ \begin{array}{c}\text{fibration}\\ \text{into conics}\end{array} & \pi & \downarrow & & \downarrow & & \begin{array}{c}\text{locally trivial}\\ \mathbb{P}^2-\text{fibration}\end{array} \\ & & S & = & S & & \end{array}$$

Here $\dim V \geq 3$, $\dim S = \dim V - 1 \geq 2$, $\mathcal{E}$ is a locally trivial sheaf of rank 3 on $S$. The points $x \in S$ over which the conic $\pi^{-1}(x) \subset \mathbb{P}^2$ degenerates comprise

the *discriminant divisor* $D \subset S$. Assume that $V/S$ is minimal (or *standard*, see [I3,S1,S2]) in the following sense:

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} S.$$

That is, $V/S$ is a Mori fiber space, see [C2]. In particular, $V/S$ has no sections. The main question to be considered is whether $V$ has other structures of a conic bundle or not. In other words, let $\pi' \colon V' \to S'$ be another conic bundle. Is an arbitrary birational map $\chi \colon V - - \to V'$ fiber-wise or not? Here is the diagram:

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \to} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \stackrel{?}{- - \to} & S', & \end{array}$$

where the ? sign means the question above.

**Theorem 7.1.** **[S1,S2]** *If $|4K_S + D| \neq \emptyset$, then $\chi$ is always fiber-wise: there exists a birational map $\alpha \colon S - - \to S'$ such that*

$$\alpha \circ \pi = \pi' \circ \chi.$$

There was no concept of birational rigidity in 1980. Now we just say that $V/S$ is birationally rigid. In [S1,S2] it was said just that the conic bundle structure, given by definition, is unique.

Let $V/S$ be a non-trivial fibration, $\dim S \geq 1$, with rationally connected (or just uniruled) fibers. Define the group of *proper* birational self-maps setting

$$\operatorname{Bir}(V/S) = \operatorname{Bir} F_\eta \subset \operatorname{Bir} V,$$

where $F_\eta$ is the generic fiber, that is, the variety $V$ considered over the field $\mathbb{C}(S)$. In other words, birational self-maps from $\operatorname{Bir}(V/S)$ preserve the fibers.

**Definition 7.1.** The fibration $V/S$ is *birationally rigid*, if for *any* variety $V'$, *any* birational map $V - - \to V'$ and *any* moving linear system $\Sigma'$ on $V'$ there exists a proper birational self-map $\chi^* \in \operatorname{Bir}(V/S)$ such that the following diagram holds:

$$\begin{array}{ccccc} V & \stackrel{\chi^*}{- - \to} & V & \stackrel{\chi}{- - \to} & V' \\ \Sigma^* & \longleftarrow & \Sigma & \longleftarrow & \Sigma' \\ c(\Sigma^*) & & \leq & & c(\Sigma'). \end{array}$$

The definition of superrigidity is word for word the same as in the absolute case.

**Proposition 7.1.** *Assume that $X/S$ is a Fano fibration with $X$, $S$ smooth such that*

$$\operatorname{Pic} X = \mathbb{Z}K_X \oplus \pi^* \operatorname{Pic} S$$

*and for any effective class $D = mK_X + \pi^*T$ the class $NT$ is effective on $S$ for some $N \geq 1$. Assume furthermore that $X/S$ is birationally rigid. Then for any rationally connected fibration $X'/S'$ and any birational map*

$$\chi \colon X - - \rightarrow X'$$

*(provided that such maps exist) there is a rational dominant map*

$$\alpha \colon S - - \rightarrow S'$$

*making the following diagram commutative:*

$$\begin{array}{ccccc} & X & \overset{\chi}{- - \rightarrow} & X' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & S & \overset{\alpha}{- - \rightarrow} & S'. & \end{array}$$

**Conjecture 7.1.** *If the fibration $V/S$ as above is sufficiently twisted over the base $S$, then $V/S$ is birationally rigid.*

We prefer not to be formal here about this conjecture. Instead of explaining what precisely is meant by the twistedness assumption, we just give an example that illustrates the point. This example has already been generalized in higher dimensions [P7].

Let us consider three-folds fibered into cubic surfaces:

$$\begin{array}{ccccc} & V & \hookrightarrow & \mathbb{P}(\mathcal{E}) & \\ \text{fibers are cubic surfaces} \quad \pi & \downarrow & & \downarrow & \text{locally trivial } \mathbb{P}^3-\text{fibration} \\ & \mathbb{P}^1 & = & \mathbb{P}^1. & \end{array}$$

Here $\operatorname{rk}\mathcal{E} = 4$, $\operatorname{Pic}\mathbb{P}(\mathcal{E}) = \mathbb{Z}L \oplus \mathbb{Z}G$, where $L$ is the class of the tautological sheaf, $G$ is the class of a fiber, and

$$V \sim 3L + mG$$

is a smooth sufficiently general divisor in the linear system $|3L + mG|$. Assuming that $3L + mG$ is an ample class, we get by the Lefschetz theorem that

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F,$$

where $F = G|_V$ is the class of a fiber. Furthermore,

$$A^2 V = \mathbb{Z}s \oplus \mathbb{Z}f,$$

where $s$ is the class of some section and $f$ is the class of a line in a fiber.

**Theorem [P4].** *Assume that $K_V^2$ does not lie in the interior of the cone of effective curves in $A^2V$. Then $V/\mathbb{P}^1$ is birationally rigid.*

**Remark.** The group of proper birational self-maps $\mathrm{Bir}(V/\mathbb{P}^1)$ was described (generators and relations) by Yu.I.Manin, see [M3]. It is very big.

**Example.** Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 4}$, so that $\mathbb{P}(\mathcal{E}) = \mathbb{P}^3 \times \mathbb{P}^1$. The variety $V$ is a divisor of bidegree $(3, m)$. It is easy to check that for $m \geq 3$ the $K^2$-condition is satisfied, so that by the theorem $V/\mathbb{P}^1$ is birationally rigid.

If $m = 2$, then the projection

$$p: V \to \mathbb{P}^3$$

onto the first factor is of degree two. Therefore there exists a Galois involution

$$\tau \in \mathrm{Bir}\, V,$$

which by construction can not be fiber-wise. Using the method of [P4], it was proved in [Sob] that the group $\mathrm{Bir}\, V$ is isomorphic to the *free* product

$$\mathrm{Bir}\, V = \mathrm{Bir}\, F_\eta * \langle \tau \rangle$$

and $V/\mathbb{P}^1$ is "almost rigid" in the following sense: each pencil of rational surfaces on $V$ can be transformed into the pencil of fibers $|F|$ by means of a birational self-map. There are no conic bundle structures on $V$. In particular, $V$ is non-rational. (Another example of "almost rigidity" see in [Gr].)

Combining the $m \geq 3$ and $m = 2$ cases, we get a far-reaching generalization of the remarkable theorem of late Fabio Bardelli [B], obtained by means of the Clemens degeneration method.

The above-described results are quite precise: all the cases $m \geq 2$ are embraced, while if $m = 1$, then $V$ is clearly rational.

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